

Optimal Pension Management under Stochastic Interest Rates, Wages, and Inflation*

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Abstract

We consider a stochastic model for a defined-contribution pension fund in continuous time. In particular, we focus on the portfolio problem of a fund manager who wants to maximize the expected utility of his terminal wealth in a complete financial market with stochastic interest rate. The fund manager must cope with two background risks: the salary risk and the inflation risk. We find a closed form solution for the asset allocation problem and so we are able to analyse in detail the behaviour of the optimal portfolio with respect to salary and inflation. Finally, a numerical simulation is presented.

JEL Classification: C61, G11, G23.

Keywords: defined-contribution pension plan, salary risk, inflation risk, stochastic optimal control, Hamilton-Jacobi-Bellman equation.

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1 Introduction

There are two extremely different ways to manage a pension fund. On the one hand, we find defined-benefit plans (hereafter DB), where benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. On the other hand, there are defined-contribution plans (hereafter DC), where contributions are fixed and benefits depend on the returns on fund's portfolio. In particular, DC plans allow contributors to know, at each time, the value of their retirement accounts. Historically, fund managers have mainly proposed DB plans, which are definitely preferred by workers. In fact, in the case of DB plans, the associated financial risks are supported by the plan sponsor rather than by the individual member of the plan. Nowadays, most of the proposed pension plans are based on DC schemes involving a considerable transfer of risks to workers. Accordingly, DC pension funds provide contributors with a service of saving management, even if they do not guarantee any minimum performance. As we have already highlighted, only contributions are fixed in advance, while the final retirement account fundamentally depends on the administrative and financial skill of the fund managers. Therefore, an efficient financial management is essential to gain contributors' trust.

The goal of the fund manager is to invest the accumulated wealth in order to optimize the expected value of a suitable terminal utility function. The classical dynamic optimization model, initially proposed by Merton (1971), assumes a market structure with constant interest rate. In the case of pension funds, the optimal asset-allocation problem involves quite a long period, generally from 20 to 40 years. It follows that the assumption of constant interest rates is not fit for our purpose. Moreover, the fund manager must cope not only with financial risks, but also with background risks. Here, by "financial risks" we mean the risks involved by the financial market, and by "background risks" we mean all the risks outside the financial market (e.g. salary and inflation). In particular, the introduction in the optimal portfolio problem of wage income causes several computational difficulties, although the underlying methodological approach is the same as that used for the no-wage income case. Merton (1969, 1971, 1990), Duffie (1996), and Karatzas and Shreve (1998) provide general treatment of optimal portfolio choice in continuous-time, without any background risk. Actually, when the background risk is considered, the stochastic partial differential equation characterising the optimal control problem, becomes harder and harder to solve. Nevertheless, since our goal is to analyse the optimal portfolio

strategies for a DC pension fund during the accumulation phase, we cannot overlook the leading role of the salary process.

In this work, we extend the stochastic model for pension fund dynamics presented in Battocchio and Menoncin (2002), and, in particular, we consider the following framework: (i) the interest rate follows an Ornstein-Uhlenbeck process (Vasiček's model, 1977), (ii) the financial market consists of three assets: a riskless asset, a stock, and a bond, which can be bought and sold without incurring any transaction costs or restriction on short sales, and (iii) we take into account two stochastic processes describing the behaviour of salaries and of the consumption price index. The reader is referred to Cox, Ingersoll, and Ross (1985) for two particular functional forms which can be used for modeling inflation. Here, we closely follow their approach.

In order to characterise the accumulation phase of the DC pension fund, we consider the case of a shareholder who, at each period $t \in [0, T]$, contributes a constant proportion of his salary to a personal pension fund. At the time of retirement T , the accumulated pension fund will be converted into an annuity.

Similar models have been recently presented by Blake, Cairns, and Dowd (2000), Boulier, Huang, and Taillard (2001) and Deelstra, Grasselli, and Koehl (2001). Especially, Blake et al. (2000) assume a stochastic process for salary including a non-hedgeable risk component and focus on the replacement ratio as the central measure for determining the pension flow. Boulier et al. (2001) assume a deterministic process for salary and consider a guarantee on the benefits. Accordingly, they strongly support the real need for a downside protection of contributors who are more directly exposed to the financial risk borne by the pension fund. Also Deelstra et al. (2001) allow for a minimum guarantee in order to minimize the randomness of the retirement account, but they describe the contribution flow through a non-negative, progressive measurable, and square-integrable process. A recent model for a DC pension scheme in discrete time is proposed by Haberman and Vigna (2001). In particular, they study both the "investment risk", that is the risk of incurring a poor investment performance during the accumulation phase of the fund, and the "annuity risk", that is the risk of purchasing an annuity at retirement in a particular recessionary economic scenario involving a low conversion rate.

The problem of optimal portfolio choice for a long-term investor in presence of wage income is also treated by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2002), and Franke, Peterson, and Stapleton (2001). Under a complete market with a constant interest rate, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization

problem for an economic agent endowed with a stochastic insurable stream of labor income. Thus, they assume that the income process does not involve a new source of uncertainty. Campbell and Viceira (2002) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyse the impact of the resolution of the labor income uncertainty on portfolio choice. They show how the investor's portfolio strategy changes when his labor income uncertainty is resolved earlier or later in life.

The methodological approach we use to solve the optimal asset-allocation problem of a pension fund is the stochastic dynamic programming. Alternative approaches (see for instance Deelstra et al. (2001), and Lioui and Poncet (2001)) are based on the Cox and Huang (1991) methodology (the so called martingale approach), where the resulting partial differential equation is often simpler to solve than the Hamilton-Jacobi-Bellman equation coming from the dynamic programming. We just underline that in this work we are able to reach the same qualitative results as Lioui and Poncet even if they do not consider any inflation risk.

In this work we present a particular case of the more general framework developed in Menoncin (2002). In fact, the author computes the optimal portfolio composition when both a stochastic background risk and a stochastic inflation risk are present. Without specifying any particular functional form for the stochastic variables involved in the problem, he offers an exact solution when the financial market is complete.

Here, in order to present a numerical simulation of this result, we specify the behaviour of the considered stochastic variables by using the most common functional forms adopted in the literature (see for instance Deelstra, Grasselli, and Koehl, 2001, and Boulier, Huang, and Taillard, 2001). By following this way, we are able to analyse in detail how the risk involved by the stochastic behaviour of salary and inflation affects the optimal portfolio composition.

First, we show that the optimal portfolio is formed by three components: (i) a preference-free hedging component depending only on the diffusion terms of assets and background variables, (ii) a speculative component proportional to both the portfolio Sharpe ratio and the inverse of the Arrow-Pratt risk aversion index, and (iii) a hedging component depending on the state variable parameters. Furthermore, after working out the expected values characterizing the solution, the optimal portfolio can be simplified to the sum of only two components: one depending on the time horizon T , and the other one independent of T . In particular, the optimal portfolio real

composition turns out to have an absolutely time independent component. Moreover, the risk aversion parameter determines if the portfolio is more or less affected by the time-dependent real component. The higher the risk aversion, the more the time-dependent real component affects the optimal portfolio. Accordingly, low values of the risk aversion parameter determine a real portfolio composition that becomes approximately constant through time.

Finally, a numerical simulation is presented in order to investigate more closely the dynamic behaviour of optimal portfolio strategy. In particular, consistently with the conventional wisdom, we show that: (i) the wealth percentage that must be invested in the stock decreases through time (from an initial level close to 73%, to a level lower than 47%), (ii) on the opposite, the wealth percentage invested in the riskless asset increases through time (from an initial level close to 3%, to a level of about 56%), (iii) the optimal percentage invested in the bond decreases (from an initial level close to 24%, to a level of about -3%). This means that the fund manager must have a more aggressive investment strategy in order to accumulate higher revenues during the first period, while he can reduce the portfolio riskiness as the retirement approaches. Moreover, we recall that the bond, at its expiration date, gives the right to receive a fixed amount of money (generally its nominal value). This means that the amount of wealth invested in the bond at the beginning of the accumulation phase must be relatively high because it may guarantee a flow of money. On the contrary, when the time horizon approaches T , the need of a certain flow becomes weaker and, very close to T , the amount of money invested in the bond can become even negative.

The work is organized as follows. Section 2 presents the general framework and exposes the financial market structure, the stochastic processes describing the behaviour of asset prices, the background risks (i.e. salaries and inflation), and the fund's wealth. Section 3 presents the stochastic optimal control problem and the main results. The optimal portfolio allocation is computed, and an explicit solution to the dynamic stochastic problem is derived. In Section 4, we show some important properties of the optimal portfolio. In Section 5, we presents a numerical simulation. Section 6 concludes.

2 The Model

In this section we introduce the market structure under which the optimal asset allocation problem is defined. We define the stochastic dynamics of the interest rate and the asset values, and we present the stochastic processes describing the behaviour of the two background risks: salaries and inflation.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, where $T > 0$ denotes the retirement time of a representative shareholder. The uncertainty involved by the financial market is described by two standard and independent Brownian motions $W_r(t)$ and $W_S(t)$, with $t \in [0, T]$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, $\mathcal{F} = \{\mathcal{F}(t)\}_{t \in [0, T]}$ is the filtration generated by the Brownian motions and \mathbb{P} represents the historical probability measure. The filtration $\mathcal{F}(t)$ can be interpreted as the information set available to the investor at time t .

The independence hypothesis on $W_r(t)$ and $W_S(t)$ implies no loss of generality since we can always shift from uncorrelated to correlated Wiener processes (and vice versa) via the Cholesky decomposition of the correlation matrix.

2.1 The Financial Market

We assume that the instantaneous riskless interest rate $r(t)$ follows an Ornstein-Uhlenbeck process (see the model of Vasiček, 1977). Then, under the historical probability measure \mathbb{P} , the process $r(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dr(t) &= \alpha(\beta - r(t))dt + \sigma dW_r(t), \\ r(0) &= r_0, \end{aligned} \tag{1}$$

where α , β , and σ are strictly positive constants. Thus, the interest rate presents a mean-reverting effect where the parameter β is the "mean" level attracting the interest rate while the strength of this attraction is measured by the parameter α .

Given the differential equation of the interest rate we can derive both its value and the value of a zero coupon bond with fixed maturity. In particular, the reader is referred to Vasicek (1977) for the demonstration of the following proposition.

Proposition 1 *Suppose that the interest rate $r(t)$ satisfies the stochastic differential equation (1), then:*

1. *the explicit solution of (1) is*

$$r(t) = (r_0 - \beta) e^{-\alpha t} + \beta + \sigma \int_0^t e^{-\alpha(t-u)} dW_r(u), \quad (2)$$

2. *the price of a zero coupon bond with maturity $\tau > t$ is given by*

$$B(t, \tau, r) = e^{b(t, \tau) - a(t, \tau)r(t)},$$

where

$$a(t, \tau) = \frac{1 - e^{-\alpha(\tau-t)}}{\alpha}, \quad (3)$$

$$b(t, \tau) = -R(\infty)(\tau - t) + a(t, \tau) \left[R(\infty) - \frac{\sigma^2}{2\alpha^2} \right] + \frac{\sigma^2}{4\alpha^3} (1 - e^{-2\alpha(\tau-t)}),$$

$R(\infty) = \beta + \frac{\sigma\lambda}{\alpha} - \frac{\sigma^2}{2\alpha^2}$ represents the return of a zero coupon bond with maturity equal to infinity, and λ denotes the constant market price of risk.

The fund manager can invest in three assets characterized by the following processes.

1. A riskless asset whose price process $S^0(t, r)$ is given by

$$\begin{aligned} \frac{dS^0(t)}{S^0(t)} &= r(t)dt, \\ S^0(0) &= S_0^0, \end{aligned} \quad (4)$$

where the dynamics of $r(t)$, under the real probability measure \mathbb{P} , is defined in Equation (1). The riskless asset can be interpreted as a bank account, paying the instantaneous interest rate $r(t)$ without any default risk.

2. A stock whose price $S(t, r)$ satisfies the following stochastic differential equation:

$$\begin{aligned} \frac{dS(t, r)}{S(t, r)} &= \mu_S(t, r)dt + \nu\sigma dW_r(t) + \sigma_S dW_S(t), \\ S(0) &= S_0, \end{aligned} \quad (5)$$

where $\nu \neq 0$ represents a volatility scale factor measuring how the interest rate volatility affects the stock volatility, and $\sigma_S \neq 0$ is the stock own volatility, while the whole stock instantaneous volatility is given by $\sqrt{\nu^2 \sigma^2 + \sigma_S^2}$. Moreover, we assume that the instantaneous mean has the form $\mu_S(t, r) = r(t) + m_S$, where $m_S > 0$ can be interpreted as a risk premium. The parameter m_S is assumed strictly positive so that the stock return is higher than the return of the short interest rate. For the sake of simplicity, we introduce in our model only one stock, which can be interpreted as a stock market index. Nevertheless, if we allow for a complete market with a finite number of stocks, no further difficulties are added to the model because the only source of troubles is the market incompleteness.

3. A bond rolling over zero coupon bonds with constant maturity. Given the instantaneous short interest rate (2), we assume that there exists a market for zero coupon bonds for every value of $\tau \in [0, T]$. According to Proposition (1), the return of a zero coupon bond with maturity $\tau \in [0, T]$ is given by

$$\begin{aligned} \frac{dB(t, \tau, r)}{B(t, \tau, r)} &= (r(t) + a(t, \tau) \sigma \lambda) dt - a(t, \tau) \sigma dW_r(t), \\ B(\tau, \tau) &= 1. \end{aligned}$$

Actually, assuming the existence of infinite zero coupon bonds is quite unrealistic. However, since the short rate dynamics has only one source of randomness, we only need one zero coupon bond to replicate the other ones. If all bonds are regarded as derivatives of the underlying (exogenously-given) instantaneous short interest rate $r(t)$, then they are all characterized by the same market price of risk (for example see Björk, 1998). Now, when the market has specified the dynamics of a basic bond price process, say with maturity τ_K , the market has indirectly specified also its market price of risk, but we have just noted that it is the same for all bonds. Accordingly, the basic τ_K -bond and the instantaneous short interest rate $r(t)$ fully determine the price of all bonds. Thus, if $B_K(t, r)$ denotes the price of a bond with constant maturity τ_K and rolling over zero coupon bonds, then we can define:

$$\frac{dB_K(t, r)}{B_K(t, r)} = (r(t) + a_K \sigma \lambda) dt - a_K \sigma dW_r(t), \quad (6)$$

where

$$a_K = \frac{1 - e^{-\alpha \tau_K}}{\alpha}.$$

Hence, the local volatility becomes constant. As Boulier et al. (2001) point out, the following equation characterises the relationship between $B(t, \tau, r)$ and $B_K(t, r)$ through the riskless asset $S^0(t)$:

$$\frac{dB(t, \tau, r)}{B(t, \tau, r)} = \left(1 - \frac{a(t, \tau)}{a_K}\right) \frac{dS^0(t)}{S^0(t)} + \frac{a(t, \tau)}{a_K} \frac{dB_K(t, r)}{B_K(t, r)}.$$

This means that the original bond can be obtained through a suitable portfolio (i.e. a linear combination) of the riskless asset and the B_K bond.

The diffusion matrix for the considered financial market is given by:

$$\Sigma \equiv \begin{bmatrix} \nu\sigma & \sigma_S \\ -a_K\sigma & 0 \end{bmatrix},$$

and, since σ_S and σ are different from zero by hypothesis, and $a_K \neq 0$ by construction, it follows that

$$\det \Sigma = \sigma_S a_K \sigma \neq 0.$$

Since we have as many risky assets as risk sources and the diffusion matrix is invertible, the market we consider is complete.

2.2 The Defined-Contribution Process

The introduction in the optimal portfolio problem of no-capital income causes several computational difficulties, although the underlying methodological approach is the same as that used for the no-wage income case. In general, because of the presence of background risks directly affecting the wealth level (e.g. salaries), the solution of the partial differential equation (PDE) characterizing the stochastic optimal control problem becomes harder and harder to compute. However, since our goal is to analyse the optimal portfolio strategies for a DC pension fund during the accumulation phase, then we cannot overlook the leading role of the salary process.

Merton (1971), in his dynamic optimization framework, examines the effects of introducing a deterministic wage income in the consumption-portfolio problem. In the more recent literature, Boulier et al. (2001), and Deelstra et al. (2001) provide two models for DC pension funds in continuous time involving deterministic salaries. Blake et al. (2000) consider a model for DC pension funds where salaries are modeled through a stochastic process including a non-hedgeable component. Haberman and Vigna (2001) provide a model for DC pension funds in discrete time with

a fixed contribution rate. The problem of optimal portfolio choice for a long-term investor in presence of wage income is treated also by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2002), and Franke, Peterson, and Stapleton (2001). Under a complete market with a constant interest rate, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labor income. Thus, they assume that the income process does not involve a new source of uncertainty. On the opposite, we introduce in the defined-contribution process a non-hedgeable risk component. Campbell and Viceira (2002) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyse the impact of labor income uncertainty resolution on portfolio choice. They show how the portfolio strategy of an investor changes when his labor income uncertainty is resolved earlier or later in life. In particular, they add the labor income to the portfolio terminal value. In the present work, the income process enters the wealth process at each time $t \in [0, T]$. Indeed, we characterize the salary process through a stochastic differential equation. Accordingly, we will show that the optimal portfolio choice crucially depends on the uncertainty involved by salary. The introduction of stochastic salaries, instead of deterministic ones, allows us to consider the effects due to the labor income uncertainty, and in particular to its resolution over time.

This paper extends the model presented in Battocchio and Menoncin (2002). In particular, we specify the functional form for the coefficients of the diffusion processes involved in the problem. By doing so, we are able not only to derive an exact solution to the optimal portfolio problem, but also to explicitly compute the expected value characterising it.

The dynamic evolution of salaries is given by

$$\begin{aligned} \frac{dL(t)}{L(t)} &= \mu_L(t) dt + \kappa_r \sigma dW_r(t) + \kappa_S \sigma_S dW_S(t) + \sigma_L dW_\pi(t), \\ L(0) &= L_0, \end{aligned} \quad (7)$$

where κ_r and κ_S are two volatility scale factors measuring how the risk sources of interest rate and stock affect the salaries, while $\sigma_L \neq 0$ is a non-hedgeable volatility whose risk source does not belong to the set of the financial market risk sources. This non-hedgeable risk source is represented by the one-dimensional standard Brownian motion $W_\pi(t)$ which is supposed to be independent of $W_r(t)$ and $W_S(t)$. Moreover, we assume that the

instantaneous mean of salaries is such that $\mu_L(t) = r(t) + m_L$, where m_L is a real constant.

After applying the Itô's lemma to the log of $L(t)$, we can find the explicit solution of Equation (7):

$$L(t) = L_0 \exp \left(\int_0^t r(\tau) d\tau + \left(m_L - \frac{1}{2} \kappa_r^2 \sigma^2 - \frac{1}{2} \kappa_S^2 \sigma_S^2 - \frac{1}{2} \sigma_L^2 \right) t + \right. \quad (8) \\ \left. + \kappa_r \sigma W_r(t) + \kappa_S \sigma_S W_S(t) + \sigma_L W_\pi(t) \right).$$

Now, we assume that each employee puts a constant proportion γ of his salary into the personal pension fund. Then, the defined-contribution level is characterized as follows:

$$C(t) = \gamma L(t),$$

whose dynamic equation is

$$dC(t) = \gamma dL(t),$$

so, in this model, the contribution growth equals the wage growth.

2.3 The Inflation

In this paper we also take into account the inflation risk. Actually, when the portfolio problem for a pension fund is considered, then the period of time that must be analysed is too long for neglecting the consumption price behaviour. In this subsection we present the stochastic partial differential equation describing the evolution of the consumption price index. In particular, we suppose that this index (p) follows the Itô process:

$$\frac{dp(t)}{p(t)} = \mu_\pi(t) dt + \rho_r \sigma dW_r(t) + \rho_S \sigma_S dW_S(t) + \sigma_\pi dW_\pi(t), \quad (9) \\ p(0) = 1,$$

where $\mu_\pi(t) = r(t) + m_\pi$, and we have put $p(0) = 1$ without loss of generality. In fact, the price level can always be normalized. The parameters ρ_r and ρ_S are two volatility scale factors measuring how the volatility of interest rate and stock affect the price index, while $\sigma_\pi \neq 0$ is the inflation own volatility. This last parameter can be also interpreted as the non-hedgeable volatility since the risk source represented by $W_\pi(t)$ does not belong to the set of financial market risk sources.

After applying the Itô's lemma to the log of $p(t)$, we can find the explicit solution of Equation (9):

$$p(t) = \exp \left(\int_0^t r(\tau) d\tau + \left(m_\pi - \frac{1}{2} \rho_r^2 \sigma^2 - \frac{1}{2} \rho_S^2 \sigma_S^2 - \frac{1}{2} \sigma_\pi^2 \right) t + \right. \quad (10) \\ \left. + \rho_r \sigma W_r(t) + \rho_S \sigma_S W_S(t) + \sigma_\pi W_\pi(t) \right).$$

We have supposed that the price process is affected by both the risk sources of interest rate and stock. This means that, in our framework, we consider the case in which the stock index and the interest rate level can be interpreted as good inflation forecasters. Actually, there exist two different approaches in this case:

1. the stock price index can be considered as an inflation forecaster ;
2. the stock price index can be considered as a variable following the inflation level: in this case we should put in Equation (5) a term containing the stochastic differential dW_π .

In this work, we prefer to adopt the first approach because it seems more consistent with the hypothesis of efficient markets. In fact, if the financial market is efficient, then the price of an asset is given by the discounted expected value of its future payoffs (under the martingale probability measure). These pay-offs should take into account also the future inflation level. Thus, in this case, the stock index is a forecaster for inflation and not vice versa.

2.4 The Nominal and the Real Fund's Wealth

The fund's nominal wealth must verify:

$$F_N(t) = \theta_0(t) + \theta_S(t) + \theta_B(t), \quad (11)$$

where $\theta_S(t)$, $\theta_B(t)$, and $\theta_0(t)$ denote the amount of money invested in the two risky assets (i.e. the stock and the bond) and in the riskless asset respectively. After differentiating Equation (11) and recalling that θ_i , for any $i \in \{0, S, B\}$, is given by the number of asset i hold in the portfolio¹

¹The number of assets hold in the portfolio is a stochastic variable. Thus, to the product of this variable with the asset value we have to apply the Itô's formula.

times the asset value, we obtain:

$$\begin{aligned} dF_N = & \theta_0 \frac{dS^0}{S^0} + \theta_S \frac{dS}{S} + \theta_B \frac{dB}{B} + \\ & + d\theta_0 + d\theta_S \left(1 + \frac{dS}{S}\right) + d\theta_B \left(1 + \frac{dB}{B}\right). \end{aligned}$$

All the terms containing the differential of the portfolio composition ($d\theta_i$) disappear thanks to the self-financing condition. In fact, the changes in the portfolio value due to the change in the asset composition θ_i must be equal to the change in contributions. The usual self-financing condition implies that all the terms containing $d\theta_i$ must be equal to zero, or to the change in consumption, if the optimal consumption problem is considered. In our framework, this equality must hold for the change in the shareholders' contribution. Accordingly, we can write:

$$d\theta_0 + d\theta_S \left(1 + \frac{dS}{S}\right) + d\theta_B \left(1 + \frac{dB}{B}\right) = \gamma dL,$$

and, finally, we obtain

$$dF_N = \theta_0 \frac{dS^0}{S^0} + \theta_S \frac{dS}{S} + \theta_B \frac{dB}{B} + \gamma dL,$$

which can be written as

$$\begin{aligned} dF_N = & (\theta_0 r + \theta_S \mu_S + \gamma L \mu_L + \theta_B (r + a_K \sigma \lambda)) dt + \\ & + (\theta_S \nu \sigma - \theta_B a_K \sigma + \gamma L \kappa_r \sigma) dW_r + \\ & + (\theta_S \sigma_S + \gamma L \kappa_S \sigma_S) dW_S + \gamma L \sigma_L dW_\pi. \end{aligned}$$

Now, we know that the real wealth can be defined as the ratio between the nominal fund's wealth and the consumption price level. Thus, we have:

$$F = \frac{F_N}{p},$$

and, after applying the Itô's lemma and substituting the value of F_N given in Equation (11), we can write:²

²We recall that the Jacobian of the real wealth is:

$$\nabla_{F_N, p} F = \begin{bmatrix} \frac{1}{p} \\ -\frac{F_N}{p^2} \end{bmatrix},$$

while its Hessian is:

$$\nabla_{F_N, p}^2 F = \begin{bmatrix} 0 & -\frac{1}{p^2} \\ -\frac{1}{p^2} & 2\frac{F_N}{p^3} \end{bmatrix}.$$

$$dF = (\theta' M + u) dt + (\theta' \Gamma' + \Lambda') dW, \quad (12)$$

where,

$$\begin{aligned} \theta &\equiv [\theta_0 \quad \theta_S \quad \theta_B]', \\ M &\equiv \frac{1}{p} \begin{bmatrix} -m_\pi + \rho_r^2 \sigma^2 + \rho_S^2 \sigma_S^2 + \sigma_\pi^2 \\ m_S - m_\pi + \rho_r^2 \sigma^2 (1 - \nu) + \rho_S \sigma_S^2 (\rho_S - 1) + \sigma_\pi^2 \\ a_K \sigma \lambda - m_\pi + \rho_r \sigma^2 (\rho_r + a_K) + \rho_S^2 \sigma_S^2 + \sigma_\pi^2 \end{bmatrix}, \\ u &\equiv \frac{\gamma}{p} L (m_L - m_\pi + \rho_r \sigma^2 (\rho_r - \kappa_r) + \rho_S \sigma_S^2 (\rho_S - \kappa_S) + \sigma_\pi (\sigma_\pi - \sigma_L)), \\ \Gamma' &\equiv \frac{1}{p} \begin{bmatrix} -\rho_r \sigma & -\rho_S \sigma_S & -\sigma_\pi \\ \sigma (\nu - \rho_r) & \sigma_S (1 - \rho_S) & -\sigma_\pi \\ -\sigma (a_K + \rho_r) & -\rho_S \sigma_S & -\sigma_\pi \end{bmatrix}, \\ \Lambda &\equiv \frac{\gamma}{p} L [\sigma (\kappa_r - \rho_r) \quad \sigma_S (\kappa_S - \rho_S) \quad \sigma_L - \sigma_\pi]', \\ W &\equiv [W_r \quad W_S \quad W_\pi]'. \end{aligned}$$

We underline that the new diffusion matrix for the financial market is given by Γ which must be invertible if we want this market to be complete. In this case, we have:

$$\det(\Gamma) = -\frac{1}{p^3} a_K \sigma \sigma_\pi \sigma_S,$$

which is different from zero because σ , σ_S , and σ_π are different from zero by hypothesis, while $a_K \neq 0$ by construction (see Equation (3)). Thus, the financial market is complete even after the introduction of the inflation risk. In fact, the inflation increases the number of risk sources, but also the number of risky assets is increased by one, due to the change of the riskless asset into a risky asset.

3 The Optimal Asset Allocation Problem

We just recall the market structure:

$$\left\{ \begin{array}{l} dr(t) = \alpha(\beta - r(t))dt + \sigma dW_r(t), \\ dS^0(t) = S^0(t)r(t)dt, \\ \frac{dS(t,r)}{S(t,r)} = \mu_S(t,r)dt + \nu\sigma dW_r(t) + \sigma_S dW_S(t), \\ \frac{dB_K(t,r)}{B_K(t,r)} = (r(t) + a_K\sigma\lambda)dt - a_K\sigma dW_r(t), \end{array} \right. \quad (13)$$

$$\frac{dL(t)}{L(t)} = \mu_L(t)dt + \kappa_r\sigma dW_r(t) + \kappa_S\sigma_S dW_S(t) + \sigma_L dW_\pi(t),$$

$$\frac{dp(t)}{p(t)} = \mu_\pi(t)dt + \rho_r\sigma dW_r(t) + \rho_S\sigma_S dW_S(t) + \sigma_\pi dW_\pi(t).$$

The goal of the fund manager is to choose a portfolio strategy in order to maximize the expected value of a terminal utility function. The argument of this utility function is the real fund's wealth. We assume an exponential utility function of the form

$$U(F) = \eta e^{\delta F},$$

where, in order to have an increasing and concave utility function, η and δ are strictly negative parameters.

This exponential (CARA) utility function is consistent with a separability hypothesis on the value function solving the dynamic problem. Since Merton (1971), the separability result has been widely shown in very different cases and it has been generally associated with a CRRA utility function. When both a stochastic background risk and a stochastic inflation are considered, Menoncin (2002) has demonstrated that the CARA (exponential) utility function is the only one consistent with a separability hypothesis on the value function.

3.1 The Stochastic Optimal Control Problem

We may formally state the stochastic optimal control problem as follows:

$$\begin{cases} \max_{\theta} \mathbb{E}_0 [\eta e^{\delta F(T)}] \\ d \begin{bmatrix} z \\ F \end{bmatrix} = \begin{bmatrix} \mu_z \\ \theta' M + u \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ \theta' \Gamma' + \Lambda' \end{bmatrix} dW, \\ z(0) = z_0, \quad F(0) = F_0, \quad \forall 0 \leq t \leq T, \end{cases} \quad (14)$$

where,

$$\begin{aligned} z_{3 \times 1} &\equiv \begin{bmatrix} r & L & p \end{bmatrix}', \\ \mu_z_{3 \times 1} &\equiv \begin{bmatrix} \alpha(\beta - r) & L\mu_L & p\mu_\pi \end{bmatrix}', \\ \Omega'_{3 \times 3} &\equiv \begin{bmatrix} \sigma & 0 & 0 \\ L\kappa_r\sigma & L\kappa_S\sigma_S & L\sigma_L \\ p\rho_r\sigma & p\rho_S\sigma_S & p\sigma_\pi \end{bmatrix}, \end{aligned}$$

and the vector z contains all the state variables but the wealth.

The Hamiltonian corresponding to Problem (14) is as follows:

$$\begin{aligned} \mathcal{H} = & \mu'_z J_z + J_F (\theta' M + u) + \frac{1}{2} \text{tr} (\Omega' \Omega J_{zz}) + (\theta' \Gamma' + \Lambda') \Omega J_{zF} + \\ & + \frac{1}{2} J_{FF} (\theta' \Gamma' \Gamma \theta + 2\theta' \Gamma' \Lambda + \Lambda' \Lambda), \end{aligned} \quad (15)$$

where $J(F, z, t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation (see Section 3.2), and verifying:

$$J(F, z, t) = \sup_{\theta} \mathbb{E}_t [\eta e^{\delta F(T)}],$$

and the subscripts on J indicate the partial derivatives.

The system of the first order conditions on \mathcal{H} is:³

$$\frac{\partial \mathcal{H}}{\partial \theta} = J_F M + \Gamma' \Omega J_{zF} + J_{FF} (\Gamma' \Gamma \theta + \Gamma' \Lambda) = 0,$$

³The second order conditions hold if the Hessian matrix of \mathcal{H} :

$$\frac{\partial^2 \mathcal{H}}{\partial \theta' \partial \theta} = J_{FF} \Gamma' \Gamma,$$

is negative definite. Since $\Gamma' \Gamma$ is a quadratic form it is always positive definite and so the second order conditions are satisfied if and only if $J_{FF} < 0$, that is if the value function is concave in F . The reader is referred to Stockey and Lucas (1989) for the assumptions that must hold on the objective function for having a strictly concave value function.

from which we obtain the optimal portfolio composition:

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - \frac{J_F}{J_{FF}}(\Gamma'\Gamma)^{-1}M - \frac{1}{J_{FF}}(\Gamma'\Gamma)^{-1}\Gamma'\Omega J_{zF},$$

and, since the matrix Γ is invertible, it becomes

$$\theta^* = \underbrace{-\Gamma^{-1}\Lambda}_{\theta_{(1)}^*} - \underbrace{\frac{J_F}{J_{FF}}(\Gamma'\Gamma)^{-1}M}_{\theta_{(2)}^*} - \underbrace{\frac{1}{J_{FF}}\Gamma^{-1}\Omega J_{zF}}_{\theta_{(3)}^*}. \quad (16)$$

Thus, we can state the following result.

Proposition 2 *Under market structure (13), the portfolio composition maximizing the investor's terminal real wealth (thus solving Problem (14)) is formed by three components: (i) a preference-free hedging component ($\theta_{(1)}^*$) depending only on the diffusion terms of assets and background variables, (ii) a speculative component ($\theta_{(2)}^*$) proportional to both the portfolio Sharpe ratio and the inverse of the Arrow-Pratt risk aversion index, and (iii) a hedging component ($\theta_{(3)}^*$) depending on the state variable parameters.*

The preference free portfolio component has an important characteristic: it minimizes the instantaneous variance of the wealth differential. In fact, from Equation (12) we can see that the variance of the growth in the investor's wealth is given by

$$\text{Var}(dF) = (\theta'\Gamma'\Gamma\theta + 2\theta'\Gamma'\Lambda + \Lambda'\Lambda) dt,$$

from which we can immediately formulate the following result.⁴

Proposition 3 *The preference-free component ($\theta_{(1)}^*$) of optimal portfolio (solving Problem (14)) minimizes the instantaneous variance of the wealth differential.*

⁴We underline that the second derivative of $\text{Var}(dF)$ with respect to θ is:

$$2\Gamma'\Gamma,$$

which is always positive definite because $\Gamma'\Gamma$ is a quadratic form.

For the second portfolio component $\theta_{(2)}^*$, we just outline that it increases when the real returns on assets (M) increase and decreases when the risk aversion ($-J_{FF}/J_F$) or the asset variance ($\Gamma'\Gamma$) increase. From this point of view, we can argue that this component of the optimal portfolio has just a speculative role.

The third part $\theta_{(3)}^*$ is the only optimal portfolio component explicitly depending on the diffusion terms of the state variables (Ω). Thus, while $\theta_{(1)}^*$ covers the investor from the risk "outside" the financial market (the so-called background risk), $\theta_{(3)}^*$ covers the investor also from the risk "inside" the financial market. We will investigate the precise role of this component after computing the functional form of the value function (see Section 3.3).

3.2 An Exact Solution

For studying the exact role of the portfolio components $\theta_{(2)}^*$ and $\theta_{(3)}^*$ (see Equation (16)), we need to compute the value function $J(F, z, t)$. It can be demonstrated (see Menoncin, 2002) that, given the exponential utility function, the value function is separable by product in wealth and in the other state variables according to the following form:

$$J(z, F, t) = \eta e^{\delta F + h(z, t)}.$$

Thus, after substituting into the Hamiltonian (15) both the functional form for the value function $J(F, z, t)$ and the optimal value of θ (see Equation (16)), the HJB equation can be simplified as follows:

$$\begin{cases} h_t + (\mu'_z - M'\Gamma^{-1}\Omega) h_z + b(z, t) + \frac{1}{2}tr(\Omega'\Omega h_{zz}) = 0, \\ h(z, T) = 0. \end{cases} \quad (17)$$

where,

$$b(z, t) \equiv \delta u - \delta M'\Gamma^{-1}\Lambda - \frac{1}{2}M'(\Gamma'\Gamma)^{-1}M.$$

Finally, we compute the function $h(z, t)$ solving Equation (17). Since we can apply the Feynman-Kač theorem to Equation (17), we can state the following result.

Proposition 4 *Under market structure (13), the portfolio composition maximizing the investor's terminal exponential utility function is as follows:*

$$\theta^* = -\Gamma^{-1}\Lambda - \frac{1}{\delta}(\Gamma'\Gamma)^{-1}M - \frac{1}{\delta}\Gamma^{-1}\Omega \int_t^T \frac{\partial}{\partial z_t} \mathbb{E}_t[b(\tilde{z}_s, s)] ds,$$

where

$$\begin{aligned} d\tilde{z}_s &= (\mu_{\tilde{z}} - \Omega' \Gamma'^{-1} M) ds + \Omega(\tilde{z}_s, s)' dW, \\ \tilde{z}_t &= z_t, \\ b(z, t) &\equiv \delta u - \delta \Lambda' \Gamma'^{-1} M - \frac{1}{2} M' (\Gamma' \Gamma)^{-1} M. \end{aligned}$$

We underline that, in our market structure, the quadratic term $M' (\Gamma' \Gamma)^{-1} M$ does not depend on the state variables. Thus, its derivative with respect to z_t is zero and so Proposition (4) can be restated in the following way:

Proposition 5 *Under market structure (13), the portfolio composition maximizing the investor's terminal exponential utility function is as follows:*

$$\theta^* = -\Gamma^{-1} \Lambda - \frac{1}{\delta} (\Gamma' \Gamma)^{-1} M - \Gamma^{-1} \Omega \int_t^T \frac{\partial}{\partial z_t} \mathbb{E}_t [u - \Lambda' \Gamma'^{-1} M] ds, \quad (18)$$

where

$$\begin{aligned} d\tilde{z}_s &= (\mu_{\tilde{z}} - \Omega' \Gamma'^{-1} M) ds + \Omega(\tilde{z}_s, s)' dW, \\ \tilde{z}_t &= z_t. \end{aligned}$$

In the next subsection we compute the expected value characterizing the third optimal portfolio component. In this way we will be able to determine how the time horizon T affect the optimal portfolio composition.

3.3 The Third Component of Optimal Portfolio

It is quite interesting to show how to compute the algebraic form for the third optimal portfolio component $\theta_{(3)}^*$. In particular, the argument of the expected value in Proposition (5) is given by:

$$u - \Lambda' \Gamma'^{-1} M = \gamma \frac{L}{p} q,$$

where q is a combination of constant parameters and does not depend on the state variables r , L , and p . Actually, its value is:

$$q \equiv -\kappa_S m_S - \frac{\sigma_L}{\sigma_\pi} (m_\pi - \rho_S m_S) + m_L + (\kappa_r - \kappa_S \nu) \sigma \lambda - \frac{\sigma_L \sigma \lambda}{\sigma_\pi} (\rho_r - \rho_S \nu).$$

Accordingly, the derivative of the expected value in Equation (18) can be written as follows:

$$\begin{bmatrix} \frac{\partial}{\partial r(t)} \mathbb{E}_t [u - \Lambda' \Gamma'^{-1} M] \\ \frac{\partial}{\partial L(t)} \mathbb{E}_t [u - \Lambda' \Gamma'^{-1} M] \\ \frac{\partial}{\partial p(t)} \mathbb{E}_t [u - \Lambda' \Gamma'^{-1} M] \end{bmatrix} = \gamma q \begin{bmatrix} 0 \\ \frac{\partial}{\partial L(t)} \mathbb{E}_t \left[\frac{\tilde{L}}{\tilde{p}} \right] \\ \frac{\partial}{\partial p(t)} \mathbb{E}_t \left[\frac{\tilde{L}}{\tilde{p}} \right] \end{bmatrix}.$$

The only term we have to compute is the expected value of the ratio between the modified processes of salaries and prices, that is to say the modified real contribution.

Firstly, we carry out the necessary computations for the modified processes of L and p . In particular, we have to compute the matrix product:

$$\Omega' \Gamma'^{-1} M.$$

According to what we have already presented in the previous sections, we can write:

$$\Omega' \Gamma'^{-1} M = \begin{bmatrix} w_1 \\ L w_2 \\ p w_3 \end{bmatrix},$$

where w_1 , w_2 , and w_3 are constant parameters given by

$$\begin{aligned} w_1 &\equiv -\sigma\lambda - \rho_r \sigma^2, \\ w_2 &\equiv \kappa_S m_S - \frac{1}{\sigma_\pi} \rho_S \sigma_L m_S - \kappa_r \sigma\lambda + \nu \kappa_S \sigma\lambda + \frac{1}{\sigma_\pi} \rho_r \sigma_L \sigma\lambda + \\ &\quad - \frac{1}{\sigma_\pi} \rho_S \nu \sigma_L \sigma\lambda - \rho_r \sigma^2 \kappa_r - \rho_S \sigma_S^2 \kappa_S + \frac{1}{\sigma_\pi} \sigma_L m_\pi - \sigma_L \sigma_\pi, \\ w_3 &= m_\pi - \rho_r^2 \sigma^2 - \rho_S^2 \sigma_S^2 - \sigma_\pi^2. \end{aligned}$$

Thus, the modified differential of the state variables \tilde{z}_s can be written as:

$$\begin{bmatrix} d\tilde{r} \\ d\tilde{L} \\ d\tilde{p} \end{bmatrix} = \begin{bmatrix} \alpha(\beta - \tilde{r}) - w_1 \\ \tilde{r} + m_L - w_2 \\ \tilde{r} + m_\pi - w_3 \end{bmatrix} ds + \begin{bmatrix} \sigma & 0 & 0 \\ \kappa_r \sigma & \kappa_S \sigma_S & \sigma_L \\ \rho_r \sigma & \rho_S \sigma_S & \sigma_\pi \end{bmatrix} \begin{bmatrix} dW_r \\ dW_L \\ dW_\pi \end{bmatrix}.$$

All these processes have a close form solution. In particular, for $s \geq t$, the solution of the interest rate process is:

$$\tilde{r}(s) = \tilde{r}(t) e^{\alpha(t-s)} + \frac{\alpha\beta - w_1}{\alpha} \left(1 - e^{\alpha(t-s)} \right) + \sigma e^{-\alpha s} \int_t^s e^{\alpha\tau} dW_r(\tau),$$

while the solutions of the other two processes are:

$$\begin{aligned}\tilde{L}(s) = \tilde{L}(t) \exp \left(\int_t^s \tilde{r}(\tau) d\tau + \left(m_L - w_2 - \frac{1}{2} \kappa_r^2 \sigma^2 - \frac{1}{2} \kappa_S^2 \sigma_S^2 - \frac{1}{2} \sigma_L^2 \right) (s-t) + \right. \\ \left. + \kappa_r \sigma (W_r(s) - W_r(t)) + \kappa_S \sigma_S (W_L(s) - W_L(t)) + \sigma_L (W_\pi(s) - W_\pi(t)) \right),\end{aligned}$$

$$\begin{aligned}\tilde{p}(s) = \tilde{p}(t) \exp \left(\int_t^s \tilde{r}(\tau) d\tau + \left(m_\pi - w_3 - \frac{1}{2} \rho_r^2 \sigma^2 - \frac{1}{2} \rho_S^2 \sigma_S^2 - \frac{1}{2} \sigma_\pi^2 \right) (s-t) + \right. \\ \left. + \rho_r \sigma (W_r(s) - W_r(t)) + \rho_S \sigma_S (W_L(s) - W_L(t)) + \sigma_\pi (W_\pi(s) - W_\pi(t)) \right).\end{aligned}$$

>From these equations we can immediately derive the value of the modified real contribution, that is to say the ratio between \tilde{L} and \tilde{p} . In particular, the expected value of this ratio can be written as follows:

$$\mathbb{E}_t \left[\frac{\tilde{L}(s)}{\tilde{p}(s)} \right] = \frac{\tilde{L}(t)}{\tilde{p}(t)} e^{q(s-t)},$$

where the boundary condition ($\tilde{z}_t = z_t$) in Proposition (5) assures that:

$$\begin{aligned}\tilde{L}(t) &= L(t), \\ \tilde{p}(t) &= p(t).\end{aligned}$$

Accordingly, we have

$$\mathbb{E}_t \left[\frac{\tilde{L}(s)}{\tilde{p}(s)} \right] = \frac{L(t)}{p(t)} e^{q(s-t)}.$$

Thus, the integral defining the third component of the optimal portfolio is given by

$$\begin{aligned}\int_t^T \frac{\partial}{\partial z} \mathbb{E}_t [u - \Lambda' \Gamma'^{-1} M] ds &= \gamma q \begin{bmatrix} 0 \\ \int_t^T \frac{1}{p(t)} e^{q(s-t)} ds \\ \int_t^T -\frac{L(t)}{p^2(t)} e^{q(s-t)} ds \end{bmatrix} = \\ &= \left(e^{q(T-t)} - 1 \right) \frac{\gamma}{p(t)} \begin{bmatrix} 0 \\ 1 \\ -\frac{L(t)}{p(t)} \end{bmatrix}.\end{aligned}$$

Finally, we can write the third optimal portfolio component as:

$$\theta_{(3)}^*(t) = \left(1 - e^{q(T-t)} \right) \gamma \frac{1}{p(t)} \Gamma^{-1} \Omega \begin{bmatrix} 0 \\ 1 \\ -\frac{L(t)}{p(t)} \end{bmatrix},$$

from which it is evident that its absolute weight on the total optimal portfolio decreases when the time becomes closer and closer to the horizon T . In fact, when $t = T$ we have $\theta_{(3)}^*(T) = 0$.

4 The Simplified Optimal Portfolio

We outline that the following equality holds:

$$\gamma \frac{1}{p(t)} \Omega \begin{bmatrix} 0 \\ 1 \\ -\frac{L(t)}{p(t)} \end{bmatrix} = \Lambda,$$

and so, we can write the third optimal portfolio component as follows:

$$\theta_{(3)}^* = \left(1 - e^{q(T-t)}\right) \Gamma^{-1} \Lambda.$$

Now, we are able to simplify Proposition (4) again, and we obtain the following result.

Proposition 6 *Under market structure (13), the portfolio composition maximizing the investor's terminal exponential utility function is given by:*

$$\theta^* = -\frac{1}{\delta} (\Gamma' \Gamma)^{-1} M - e^{q(T-t)} \Gamma^{-1} \Lambda,$$

where:

$$q \equiv -\kappa_S m_S - \frac{\sigma_L}{\sigma_\pi} (m_\pi - \rho_S m_S) + m_L + (\kappa_r - \kappa_S \nu) \sigma \lambda - \frac{\sigma_L \sigma \lambda}{\sigma_\pi} (\rho_r - \rho_S \nu).$$

This result shows that the optimal portfolio is actually formed by two components: one depending on the time horizon T and the other one independent of T .

Furthermore, we underline that it is possible to write the matrix terms of θ^* as follows:

$$\begin{aligned} -\frac{1}{\delta} (\Gamma' \Gamma)^{-1} M &\equiv \frac{1}{\delta} p(t) \phi_1, \\ -\Gamma^{-1} \Lambda &\equiv \gamma L(t) \phi_2, \end{aligned}$$

where $\phi_1, \phi_2 \in \mathbb{R}^{3 \times 1}$ are two vectors of parameters which do not depend on time. Thus, the optimal portfolio can be written in real term as follows:

$$\frac{\theta^*}{p(t)} = \frac{1}{\delta} \phi_1 + \gamma \frac{L(t)}{p(t)} e^{q(T-t)} \phi_2,$$

from which we see that the first component of the optimal portfolio real composition is absolutely time independent. The risk aversion parameter δ determines if the portfolio is more or less affected by the time-dependent real component. The higher δ , the more the time-dependent real component affects the optimal portfolio. Accordingly, low values of δ determines a real portfolio composition that tends to be constant through time. In the next section, where we carry out a simulation of the model, we highlight the necessity of assigning a numerical value for δ which is consistent with the initial value given to the salary process. In particular, a too low absolute value of δ (with respect to L_0) leads to an optimal strategy which is practically constant through time.

Now it can be interesting to investigate which is the total amount of wealth invested in the financial assets. In particular, we have:

$$\mathbf{1}'\theta^* = \frac{1}{\delta} p(t) \mathbf{1}'\phi_1 + \gamma L(t) e^{q(T-t)} \mathbf{1}'\phi_2, \quad (19)$$

where $\mathbf{1} \in \mathbb{R}^{3 \times 1}$ is a vector containing only ones. After computing the products $\mathbf{1}'\phi_1$ and $\mathbf{1}'\phi_2$ we have:

$$\mathbf{1}'\theta^* = \frac{1}{\delta} p(t) \left(1 - \frac{1}{\sigma_\pi^2} \sigma \lambda (\rho_r - \rho_S \nu) - \frac{1}{\sigma_\pi^2} (m_\pi - \rho_S m_S) \right) + \gamma L(t) e^{q(T-t)} \left(\frac{\sigma_L}{\sigma_\pi} - 1 \right).$$

We see that the sign of the time-dependent component is determined by the ratio between the volatility terms σ_L and σ_π . In particular, since γ is always positive and $L(t)$ is positive too,⁵ if $\sigma_L > \sigma_\pi$ then the time-dependent component is positive, on the contrary it is negative.

In the numerical simulation which follows we have assumed $\sigma_\pi > \sigma_L$ because it seems more reasonable that the inflation own volatility is higher than the salaries own volatility. This means that when the time t approaches the horizon T the amount of wealth invested in the financial asset tends to increase. This is consistent with the hypothesis that the fund's wealth mainly increases thanks to the contributions. Thus, at the beginning, the amount of wealth invested in financial assets is low with respect to the contributions while at the end of the accumulation period, the financial wealth dominates the contributions.

⁵We recall that $L(t)$ is log-normally distributed and it cannot take negative values.

5 Numerical Application

In this section we set a numerical application in order to analyse the dynamic behaviour of the optimal portfolio strategy derived in Section 3. The following table reports the set of parameters characterizing the financial market, the defined-contribution process and the inflation process. In particular, the set of parameters representing the financial market is consistent with the numerical analysis presented by Boulier et al. (2001).

Interest rate		Defined-contribution process	
Mean reversion, α	0.2	Risk premium, m_L	0.01
Mean rate, β	0.05	Volatility scale factor, κ_r	0.7
Volatility, σ	0.02	Volatility scale factor, κ_S	0.9
Initial rate, r_0	0.03	Non-hedgeable volatility, σ_L	0.01
Fix-maturity bond		Initial salary, L_0	100
Maturity, τ_K	20	Contribution rate, γ	0.12
Market price of risk, λ	0.15	Inflation process	
Stock		Risk premium, m_π	-0.01
Risk premium, m_S	0.06	Volatility scale factor, ρ_r	0.9
Volatility scale factor, ν	3	Volatility scale factor, ρ_S	0.8
Stock own volatility, σ_S	0.19	Non-hedgeable volatility, σ_π	0.015

We consider a contribution period before retirement equal to 40 years. The absolute risk aversion of the investor is given by $\delta = -20$. The value we have assigned to δ is consistent with the initial value of the salary process (L_0). In fact, from Equation (19), we note that there must be a suitable trade-off between the initial value of the salary process and the scale of values characterising the risk aversion index. This allows us to avoid the case of an optimal portfolio rule practically constant through time.

The optimal proportion invested in the riskless asset increases from an initial value close to 3%, to about 56%. On the other hand, the optimal proportion invested in the two risky assets progressively decrease with respect to time. In particular, the stock proportion declines from an initial value of about 73%, to about 47%, while the proportion invested in the long-term

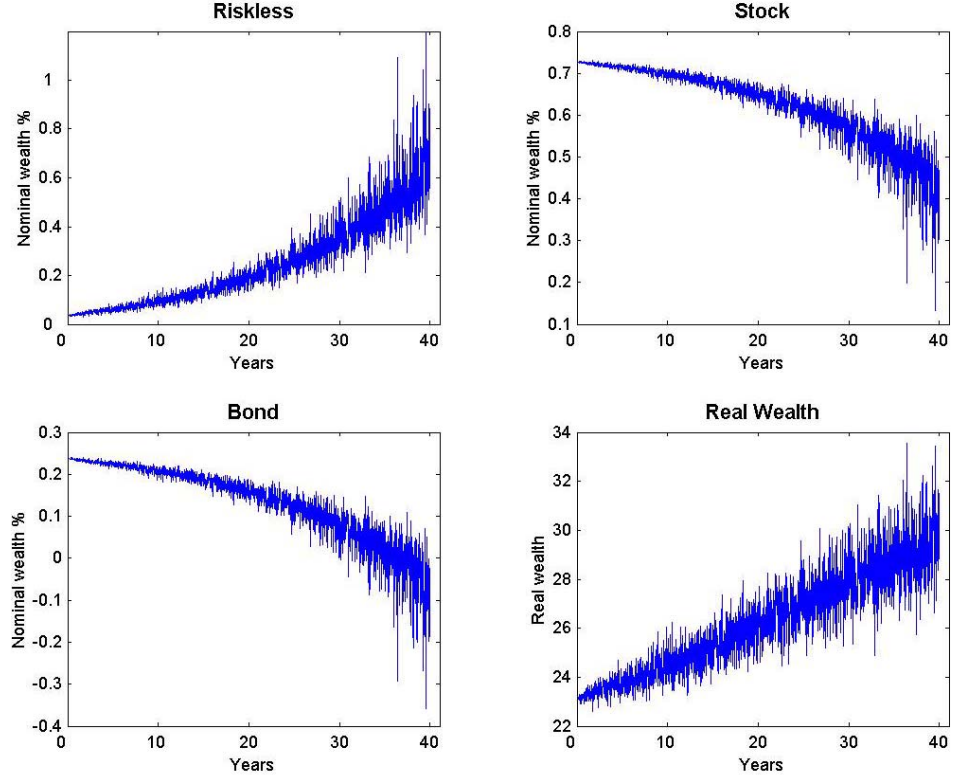


Figure 1: Percentage composition of optimal portfolio and real wealth

bond declines from an initial value close to 24%, to about -3% . The investment trends of the three assets are consistent with the portfolio managers experience and the conventional wisdom. During the beginning of the contribution period, the fund manager realizes a more aggressive investment policy in order to boost the fund. Consistently, as the time approaches the retirement in T , Figure (1) shows a shift of wealth from the investment in risky assets to the money account. However, the fund manager maintain a diversified portfolio until retirement.

Figure (1) highlights how the evolution of the optimal portfolio strategies is actually affected by the realization of the stochastic variables characterizing our economy. Consistently, the uncertainty related to the decisions which must be taken by the fund manager augment as we approach the

retirement, or better as we deviate from the present.

We recall that the bond, at its expiration date, gives the right to receive a fixed amount of money (generally its nominal value). This means that the amount of wealth invested in the bond at the beginning of the accumulation phase must be relatively high because it may guarantee a flow of money. On the contrary, when the time horizon T approaches, then this need of a certain flow become weaker and, very close to T , the amount of money invested in the bond can become even negative.

Another time, with respect to the traditional approach, we can see that the riskless asset play a residual role in the optimal portfolio composition. At the beginning of the accumulation phase, the need of an aggressive strategy for creating a higher wealth level leads to a high percentage of stock in the optimal portfolio, while the need of a guarantee for a financial flow leads to a relative high investment in the bond. Thus, the investment in the riskless asset is very low. Actually, the revenue guaranteed by the riskless asset is instantaneous, while the revenue the manager needs for guaranteeing a flow on the future can come only from bond.

While the riskiness of the strategy decreases and so the need for a guarantee at time T , both the investments in stock and bond decreases and so, the percentage of wealth invested in the riskless asset increases. Moreover, we note an increasing slope, in absolute terms, for all assets. This evidence suggests the necessity for a more frequent adjustment of the investment strategies in the last years of the accumulation phase.

In Boulier et al. (2001), the mean composition of the pension fund is characterised by deterministic trends. On the opposite, given the length of the accumulation phase and the central role of the contribution flow, we strongly support the need for a stochastic framework. In contrast with Boulier et al. (2001), who found a hefty short position in cash, our model implies persistent long position in the riskless asset. This result is consistent with the restriction which usually prevents funds from borrowing.

Concerning the practical aspects of the fund management, we plan also to characterise the optimal portfolio composition through the definition of a confidence interval.

6 Conclusion

In this paper we have analysed the optimal portfolio problem for a defined contribution pension fund maximizing the expected value of its terminal utility function. The shareholders contribute a constant percentage of their

salaries into the fund. The fund manager faces two kind of risks: the risk linked to the shareholders' salaries, which are supposed to be stochastic, and the risk linked to the inflation stochastic process. On the financial market there are a stock, a bond, and a riskless asset.

After specifying the functional form for the coefficients of the stochastic processes characterizing the model, we have found a close form solution to the asset allocation problem. We have highlighted that the inflation process changes the riskless asset into a risky asset. In fact, the original risky asset is no more able to guarantee a riskless return because it cannot hedge against the inflation risk. Accordingly, the new market structure, which maintains its completeness, is defined by three risky assets.

First, we have characterized the optimal portfolio as the sum of three components: (i) a preference-free hedging component depending only on the diffusion terms of assets and background risks, (ii) a speculative component proportional to both the portfolio Sharpe ratio and the inverse of the Arrow-Pratt risk aversion index, and (iii) a hedging component depending on the state variable parameters. Then, after working out the expected values characterising the solution, we have been able to simplify the optimal portfolio as the sum of only two new components: one depending on the time horizon T , and the other one independent of T . In particular, we have noted that the optimal portfolio real composition turns out to have an absolutely time independent component.

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